Comp. & Maths. with Appls. Vol. 11, No. 7/8, pp. 831–841, 1985
Printed in Great Britain.

0097-4943/85 $3.00 + .00

THE NATURAL REFERENCE WAVENUMBER FOR PARABOLIC APPROXIMATIONS IN OCEAN ACOUSTICS

ALLAN D. PIERCE
Georgia Institute of Technology, Atlanta, GA 30332, U.S.A.

Abstract—A natural definition of the reference wavenumber k_0 that appears in almost all underwater sound formulations based on a parabolic approximation emerges from Rayleigh's principle for progressive waves: the integrals over the depth of the kinetic- and potential-energy densities are equal. A formulation using this definition conforms to and refines Fitzgerald's suggestion that when the field is a sum of guided modes, the selected k_0 should be a weighted average of the wavenumbers for the excited modes.

1. INTRODUCTION

A variety of formulations[1–10] for constant-frequency underwater sound propagation is made up of those based on parabolic approximations; almost all require some judicious choice of a reference wavenumber k_0. The present paper shows that there is a natural choice for this parameter and gives a formulation of a parabolic equation (PE) model that is consistent with this choice.

Underlying a parabolic approximation is the concept that a wave equation's form with regard to one of its independent variables can be simplified if the propagation is primarily in the corresponding spatial direction with negligible reflection. Taking advantage of this quasi-unidirectional propagation requires, however, an estimate of a representative phase velocity in this direction. Equivalently, for constant-frequency propagation, one requires an estimate of an average rate k_0 of phase change with increasing propagation distance. Use of such a technique dates back at least as far as Korteweg and de Vries' derivation[11], published in 1895, of a simplified one-dimensional wave equation for shallow-water waves, with approximate account taken for dispersion and nonlinear steepening; the reference phase velocity was the low-frequency low-amplitude limit for waves in water of the nominal depth. Burgers' equation[12, 13] for nonlinear propagation with dissipation is another well-known older example, the reference phase velocity being the ambient sound speed. The parabolic approximation introduced by Leontovich and Fock[14] was intended for quasi-planar propagation in a homogeneous medium with weak diffraction transverse to the direction of propagation; the natural choice for k_0 was ω/c, where c is the wave speed. Similarly, there is no ambiguity in k_0 when the parabolic approximation is applied to nominally plane-wave propagation through a random medium.

If guided waves are concentrated in a depth region where the sound speed has a local minimum (as for the SOFAR channel in the ocean), and if the rays along which the energy is predominantly being carried are at low angles with regard to the channel axis, then a parabolic equation also emerges as a good approximate description of the wave field; the choice here for k_0 is $\omega/c_{\min}$, where $c_{\min}$ is the minimum sound speed.

Numerical experimentation carried out in the first few years after the original introduction (c. 1973) of parabolic approximations into underwater sound demonstrated that good approximate results could be attained even when the propagation is not limited to a SOFAR channel and when the sound speed and density profiles are not so stringently idealized. In such circumstances there is no *a priori* obvious choice for k_0, so the selection of k_0 was left to the discretion of the programmer. This arbitrariness was dismissed by many workers as being a matter of no practical concern because the results calculated with parabolic approximations are usually extremely insensitive to small changes in k_0.

Fitzgerald[15] and McDaniel[16], in papers published in 1975, pointed out that, when the environment is range independent and when the field is described by a single guided mode, a parabolic approximation becomes exact if k_0 is chosen to be the mode's horizontal wave number k_n (equal to ω divided by the phase velocity of that mode). If there is more than one mode, however, then the parabolic approximation introduces intrinsic errors. These appear as spurious phase differences, which increase linearly with range, between terms in the modal sum, so that, at sufficiently large r, one could be predicting an interference between two modes when there actually is a reinforcement. Ideally, one would like to choose k_0 so that such spurious results are postponed (to larger r) and so that the errors, both in the magnitude and phase, of the overall sum are minimal at smaller r. Since one cannot choose k_0 to equal each of the appropriate k_n for several modes simultaneously, some compromise is required for multimodal propagation. Fitzgerald[15] suggested that one choose k_0 to be the average of the k_n for those modes that are excited. Doing so, however, requires some decision as to how the various k_n's should be weighted. If two modes are present, the first very strongly excited, the second very weakly, then it seems fitting that the selected k_0 should be closer to the k_n of the first mode than to that of the second mode.

The primary principle employed here for the selection of k_0 ensues from the observation of Rayleigh[17] that constant-frequency low-amplitude wavefields progressing in one direction invariably are such that average kinetic energy equals average potential energy. As demonstrated below, the requirement that this principle apply to wavefields governed by a parabolic equation leads to a unique expression for k_0. When the modal description is applicable, the derived result is wholly consistent with Fitzgerald's suggestion and, moreover, resolves the question of how the modal wavenumbers should be weighted. Use of the result, however, does not require that one know the modal decomposition of the wavefield or that the field be describable as a sum of guided modes.

2. PARABOLIC APPROXIMATIONS FOR UNDERWATER SOUND

Parabolic approximations are used in underwater acoustics for the solution of the reduced wave equation[1, 18]

$$\rho\nabla \cdot (\rho^{-1}\nabla p) + (\omega/c)^2 p = 0, \tag{1}$$

where p is the complex amplitude of the acoustic pressure, ρ is the ambient density, c is the sound speed, and ω is the angular frequency of the source. The environmental parameters c and ρ depend primarily on depth z below the water surface and to a lesser (sometimes regarded as negligible) extent on the horizontal coordinates x and y. One usually is interested in predicting the field at moderate-to-large horizontal distances from a source of limited extent under circumstances when the wave tends to be guided by natural interfaces and by ducts in the sound-speed profile. Customary approximations assume that the intrinsic cylindrical spreading is accounted for by a factor $r^{-1/2}$, such that p is taken to be of the form

$$p = r^{-1/2}\Phi(r, z), \tag{2}$$

where at sufficiently large horizontal distances r from the source, the function $\Phi(r, z)$ satisfies

$$\rho(\rho^{-1}\Phi_r)_r + \rho(\rho^{-1}\Phi_z)_z + (\omega/c)^2\Phi = 0. \tag{3}$$

Here the r and z subscripts denote partial differentiation with respect to the corresponding coordinates.

A parabolic approximation to Eq. (3) is any approximation that should be of the form (with the time dependence assumed to be $e^{-i\omega t}$,

$$\Phi(r, z) = e^{i\chi(r)} F(r, z), \tag{4a}$$

$$\chi(r) = \int_0^r k_0 \, dr, \tag{4b}$$

where k_0 is the reference wave number (possibly depending on range r), and $F(r, z)$ satisfies a partial differential equation (parabolic equation, customarily abbreviated PE) that is first order in the range coordinate r.

The various formulations of PE models[1–10] that exist in the literature differ either in the form of the parabolic equation, the manner in which it is recast for numerical calculation, or the manner in which the medium is idealized. The fundamental form to which others can be regarded as modifications is the version originally introduced by Tappert[1]:

$$F_r = (ik_0/2)[k_0^{-2}F_{zz} + (n^2 - 1)F], \tag{5}$$

where

$$n = k_0^{-1}\,\omega/c \tag{6}$$

can be regarded as an index of refraction.

3. DERIVATION OF PARABOLIC EQUATION FROM A VARIATIONAL PRINCIPLE

The present paper's basic conceptual framework with regard to the selection of k_0 should be nearly the same regardless of what preexisting PE formulation is chosen as a point of departure. Consequently, with the hope that a clearer exposition is achieved, the discussion here is limited to a model that incorporates only simple extensions of Eq. (5); some extensions are necessary if one is to consider the possibility that k_0 might vary with range, and if one desires a version that does not preclude density gradients or discontinuities.

A parabolic equation that contains the desired modifications is quickly derived from the variational principle[19] that corresponds to Eq. (3), this being

$$\delta \int\int \left[\Phi_r\Phi_r^* + \Phi_z\Phi_z^* - \left(\frac{\omega}{c}\right)^2 \Phi\Phi^*\right] \rho^{-1}\,dr\,dz = 0, \tag{7}$$

with the understanding that ϕ and ϕ^* are independent functions. If one seeks the most natural approximate solution of the generic form (4), where F is to satisfy a partial differential equation that is first order in r and second order in z, then one must discard the term $\rho^{-1}F_rF_r^*$ in the integrand, so the approximated variational indicator becomes

$$\delta \int\int [ik_0\,(FF_r^* - F^*F_r) + F_zF_z^* + k_0^2\,(1 - n^2)FF^*]\,\rho^{-1}\,dr\,dz = 0. \tag{8}$$

A straightforward application of the techniques of the calculus of variations then yields the parabolic equation

$$(k_0\rho^{-1}F)_r + k_0\rho^{-1}F_r = i[(\rho^{-1}F_z)_z + (k_0^2/\rho)(n^2 - 1)F]. \tag{9}$$

Equivalently, one can write

$$F(r, z) = (\rho/k_0)^{1/2}\psi(r, z), \tag{10}$$

such that the pressure amplitude p is of the form

$$p = r^{-1/2}(\rho/k_0)^{1/2}e^{i\chi(r)}\psi(r, z), \tag{11}$$

where $\psi(r, z)$ satisfies the partial differential equation

$$\psi_r = (ik_0/2)[k_0^{-2}\rho^{1/2}\{\rho^{-1}(\rho^{1/2}\psi)_z\}_z + (n^2 - 1)\psi]. \tag{12}$$

A useful feature of the parabolic equation in the form of Eq. (9) is that it can be applied to environments where the density and sound-speed change discontinuously (such as at an interface between a shallow-water layer and an underlying fluid bottom), provided that one regards such discontinuities as the limiting case of a situation where the environmental parameters change rapidly but continuously over a very short distance. Pillbox derivations, such as commonly used in electromagnetic theory texts[20] to derive interface relations from Maxwell's equations, when applied to Eq. (9), yield the continuity conditions

$$C_1 = F_+(r, \zeta) = F_-(r, \zeta), \tag{13a}$$

$$C_2 = \{(F_z - ik_0F\zeta')/\rho\}_+ = \{(F_z - ik_0F\zeta')/\rho\}_-, \tag{13b}$$

$$C_3 = \{F_r + \zeta'F_z\}_+ = \{F_r + \zeta'F_z\}_- \tag{13c}$$

at any interface described generically by $z = \zeta(r)$ and having slope $\zeta'(r)$. A customary approach[21, 22] in numerical work is to take ρ as constant in each layer, to use separate parabolic equations for the individual layers, and to use continuity conditions such as those above to couple the solutions of the separate parabolic equations.

4. PHYSICAL PRINCIPLES FOR k_0 SELECTION

The present paper avoids the task of defining a "best k_0". Any criteria for what is "best" will be affected by the specific application. One could require, for example, given a certain environmental model and a certain model for wave excitation, that the error in magnitude of the pressure amplitude be minimal, if not zero, at a specified far-field point. Such a criterion may lead to a unique value of k_0, but the value would change if the specified far-field point were to be changed, or if one were to ask instead for a minimal error in phase.

A more reasonable task is to seek the most "natural choice" for k_0, this being whatever integrates the overall model, parabolic equation plus equation(s) defining k_0, into a logically self-contained physical theory that is consistent with recognized general physical principles. Although such a concept of a "natural" k_0 may seem to also be ambiguous, the ambiguity becomes progressively more and more restricted once one begins to list what should be the "recognized general physical principles" to which the theory should conform. The train of reasoning is heuristic in the sense used by Polya[23]—i.e. not regarded as final and strict but as provisional and plausible only, whose purpose is to discover a good, simple, approximate and aesthetically pleasing physical model that can subsequently be tested by experiment against more nearly exact models. Well-known examples of physical theories that developed along such lines are those of beams[24] and thin plates[25]. A yet stronger analogy can be drawn with the Timoshenko model[26] of a beam, which introduces a heuristic constant, the Timoshenko shear constant[27].

A list of desirable requirements on the overall physical theory of wave propagation subsuming a parabolic approximation includes the following:

4.1 *Huygens' principle*

The theory should be a local theory with respect to propagation out in range; Huygens' principle[28, 29] must hold in the sense that propagation is a Cauchy problem: a complete knowledge of the field (all depths) at a given range r is sufficient to predict the field at range $r + \Delta r$. A corollary of this interpretation of Huygens' principle is that the reference wave number k_0 can depend at most on the depth profiles of sound speed, density, and the field variable $F(r, z)$ at the range of current interest. It does leave open, however, the possibility that k_0 may vary with range.

4.2 *Conservation of energy*

To the extent that no sound absorption is included in the model or in the limit where incorporated absorption coefficients are set to zero, the theory must conform to the conservation of energy in the sense that the power flow is independent of range. This implies, in particular, that the defining equations for the theory should have as an exact corollary (Re implying real part)

$$\frac{d}{dr}\left\{r \int \mathrm{Re}(p^*u)\, dz\right\} = 0, \tag{14}$$

where u is whatever the model takes for the complex amplitude of the horizontal component of the fluid velocity. The natural identification consistent with the spirit of the parabolic approximation [which assumes that the range dependence is primarily contained in the exponential factor in Eq. (4a)] and with the range component of the momentum equation (Euler's equation of motion for a fluid) is

$$u = (k_0/\omega\rho)p = (k_0/\omega\rho)r^{-1/2}e^{i\chi} F. \tag{15}$$

Consequently, Eqs (2)–(4) lead to the requirement, given that k_0 is real,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left\{k_0 \int |F|^2\rho^{-1}\,\mathrm{d}z\right\} = 0. \tag{16}$$

Given that (15) is the theory's appropriate identification for the complex amplitude of the horizontal component of the fluid velocity, then the corresponding identification for that of the z-component v must be

$$v = (1/i\omega\rho)p_z = (1/i\omega\rho)r^{-1/2}\,e^{i\chi}\,F_z. \tag{17}$$

Equation (16) turns out to be a derivable consequence of the parabolic equation (9), independent of the expression for k_0. To prove that such is so, one multiplies Eq. (9) by F^*, multiplies the complex conjugate of Eq. (9) by F, then adds the two equations, obtaining

$$\begin{aligned} 2(k_0\rho^{-1}|F|^2)_r &= i[F^*(\rho^{-1}F_z)_z - F(\rho^{-1}F_z^*)_z] \\ &= i(\mathrm{d}/\mathrm{d}z)[\rho^{-1}F^*F_z - \rho^{-1}F\,F_z^*]. \end{aligned} \tag{18}$$

Integration of both sides over z, with the boundary condition that $\rho^{-1}F^*F_z$ vanish at the two endponts, subsequently yields Eq. (16).

The principal importance of Eq. (16) from the standpoint of the derivation for k_0 is that its validity supports the identifications of (15) and (17) for u and v; these are needed for the analytical development given in the next section.

4.3 *Rayleigh's principle for progressive waves*

All conservative linear-wave-types of constant frequency, regardless of the physical systems to which they correspond, appear to adhere to a general principle first announced by Rayleigh[17]: if the propagation is progressive, such that energy is being transported on the average in one direction, with no propagation in the backward direction, then the average kinetic energy in the wave must equal the average potential energy. Some averaging is, in general, required, plane sound waves in a homogeneous fluid being a notable exception. For gravity waves in water, for example, one must do an averaging of energy densities over time and an integration over depth before the equality emerges. For flexural waves in beams, one must average energies per unit beam length over time. Another statement of the principle is that the average Lagrangian density for the wave must be zero.

That this principle must apply as well to underwater acoustics can be readily demonstrated when the field is a superposition of guided modes all propagating in the same direction. Exact solution[30] of Eq. (3) for such circumstances yields

$$p = r^{-1/2}\sum_n A_n\,e^{ik_nr}Z_n(z), \tag{19a}$$

$$u = r^{-1/2}\sum_n \frac{k_n}{\omega\rho} A_n\,e^{ik_nr}Z_n(z), \tag{19b}$$

$$v = r^{-1/2}\sum_n \frac{i}{\omega\rho} A_n\,e^{ik_nr}Z_n'(z) \tag{19c}$$

for the acoustic pressure amplitude and the corresponding two components of the fluid velocity. [The latter follow from the expression for p and the linear version of Euler's equation: the derivation presumes that r is sufficiently large that the derivative of the factor $r^{-1/2}$ in Eq. (2) can be neglected in the evaluation of u.] Here the constants A_n are the modal coefficients; and the modal depth-profiles $Z_n(z)$ are the ordered eigenfunctions of a Sturm–Liouville eigenvalue problem[31] consisting of the ordinary differential equation (with appropriate homogeneous boundary conditions)

$$\rho(\rho^{-1}Z_n')' + [(\omega/c)^2 - k_n^2]Z_n = 0. \tag{20}$$

The quantity k_n^2, the square of the horizontal wave number, is the corresponding eigenvalue. (Here the primes denote differentiation with respect to the argument z.)

From the above modal sums for p, u and v, one can readily verify Rayleigh's assertion; with no additional approximations these lead to the energy statement

$$\int \frac{1}{4}\,\rho(|u|^2 + |v|^2)\,\mathrm{d}z = \int \frac{1}{4}\,(\rho c^2)^{-1}\,|p|^2\,\mathrm{d}z, \tag{21}$$

where the integrand on the left side is the time-averaged kinetic energy per unit volume, that on the right side is the time-averaged potential energy per unit volume. [In carrying out the manipulations to demonstrate that this follows from Eqs (19) and (20), one uses the orthogonality of guided modes

$$\int \rho^{-1}\,Z_n(z)Z_m(z)\,\mathrm{d}z = 0 \quad \text{if } n \neq m, \tag{22}$$

which is derivable from Eq. (20).]

Equation (21) is the keystone of the argument that leads to an expression for the natural k_0. The author refers to it as Rayleigh's principle for progressive waves, because the general statement is due to Rayleigh, and because it is strongly related to what is referred to as Rayleigh's principle[32, 33] in the vibrations literature; usually the latter is thought of as a variational principle, but a weaker statement that results from it is that, if a linear system is vibrating in a natural mode of vibration of constant frequency, then the average kinetic energy must equal the average potential energy. An analogous result is that if any conservative linear system is in a state of free vibration, not necessarily of constant frequency, then the time-averaged kinetic energy for the system as a whole must equal the time-averaged potential energy.

5. DERIVATION OF EXPRESSION FOR k_0

Once one accepts the general principles set forth in the preceding section, then the derivation of an expression for the natural reference wave number becomes straightforward. One simply inserts

$$|u|^2 = \frac{(k_0/\omega\rho)^2|F|^2}{r} \quad |v|^2 = \frac{(1/\omega\rho)^2|F_z|^2}{r} \quad |p|^2 = |F|^2/r \tag{23}$$

into Rayleigh's principle (21) for progressive waves, then solves for k_0^2, obtaining

$$k_0^2 = \frac{\int (\omega/c)^2\rho^{-1}|F|^2\,\mathrm{d}z - \int \rho^{-1}|F_z|^2\,\mathrm{d}z}{\int \rho^{-1}|F|^2\,\mathrm{d}z}, \tag{24}$$

which is the central result of this paper.

6. RANGE DEPENDENCE OF k_0

Expression (24) produces a reference wave number that will, in general, vary with range r. Nevertheless, the overall theoretical formulation is such that one can prove that k_0^2 must be independent of range if the environmental parameters are range independent. To demonstrate this and to discover what causes a range variation of k_0, one must take the derivative of Eq. (24) with respect to r. It is of interest to know how a sloping interface affects k_0, so the analysis here allows for an internal surface $z = \zeta(r)$ along which ρ and c are discontinuous. This, however, causes the three integrands that appear in Eq. (24) to be discontinuous, so one cannot blindly interchange the order, integration over z followed by differentiation with respect to r, in evaluating the range derivatives of the integrals. To avoid any mathematical pitfalls of this nature, one begins with reexpressing of Eq. (24) in the form

$$\int_0^{\zeta} I \, \mathrm{d}z + \int_{\zeta}^{\infty} I \, \mathrm{d}z = 0, \tag{25}$$

where

$$I = \frac{[k_0^2 - \omega^2/c^2]|F|^2}{\rho} + \frac{|F_z|^2}{\rho}. \tag{26}$$

The r-derivative of Eq. (25) yields

$$(I_- - I_+)\zeta' + \int_0^{\zeta} \frac{\partial I}{\partial r} \mathrm{d}z + \int_{\zeta}^{\infty} \frac{\partial I}{\partial r} \mathrm{d}z = 0, \tag{27}$$

where the subscripts, $-$ and $+$, refer to values on the lower-z and the higher-z sides of the interface. A sequence of straightforward manipulations allows one to express the derivative $\partial I/\partial r$ as

$$\frac{\partial I}{\partial r} = \left[\left(\frac{k_0^2}{\rho}\right)_r - \left(\frac{\omega^2}{\rho c^2}\right)_r\right] |F|^2 + \left(\frac{1}{\rho}\right)_r |F_z|^2 - (F_r X^* + F_r^* X) + \frac{\partial J}{\partial z}, \tag{28}$$

with the abbreviations

$$X = (F_z/\rho)_z + (k_0^2/\rho)(n^2 - 1)F, \tag{29a}$$

$$J = (F_r F_z^* + F_r^* F_z)/\rho. \tag{29b}$$

The differential equation (9) can be expressed in the form

$$F_r = -(\rho/2k_0)(k_0/\rho)_r F + (i\rho/2k_0)X, \tag{30}$$

so one obtains the equivalence

$$F_r^* X + F_r X^* = -(\rho/2k_0)(k_0/\rho)_r[\partial K/\partial z - 2I], \tag{31}$$

with the abbreviation

$$K = (F^* F_z + F F_z^*)/\rho = (1/\rho)\partial|F|^2/\partial z. \tag{32}$$

Consequently, expression (28) reduces to

$$\frac{\partial I}{\partial r} = \frac{\rho}{2}\left(\frac{1}{\rho}\right)_r \frac{\partial K}{\partial z} - \left(\frac{\omega^2}{c^2}\right)_r \frac{|F|^2}{\rho} + 2k_0 \frac{\mathrm{d}k_0}{\mathrm{d}r}\left[|F|^2\rho^{-1} - \frac{I}{2k_0^2}\right] + \frac{\partial}{\partial z}\left[\frac{1}{2k_0}\frac{\mathrm{d}k_0}{\mathrm{d}r} K + J\right]. \tag{33}$$

This in turn, when substituted into Eq. (27), with subsequent use of Eq. (25), yields

$$\frac{dk_0^2}{dr}\int |F|^2\rho^{-1}\,dz = P\int\left[\left(\frac{\omega^2}{c^2}\right)_r |F|^2\rho^{-1} - \frac{\rho}{2}\left(\frac{1}{\rho}\right)_r \frac{\partial K}{\partial z}\right]dz + (I\zeta' + J)_+ - (I\zeta' + J)_- + \frac{1}{2k_0}\frac{dk_0}{dr}\{K_+ - K_-\}, \quad (34)$$

where P implies that one takes the principal value of the indicated integral. It follows from the continuity conditions (13a,b) that K must be continuous; hence the last term in Eq. (34) is zero.

On either side of the interface, the quantity $J + \zeta' I$ that appears in Eq. (34) can be equivalently expressed, after a straightforward series of manipulations, as

$$J + \zeta' I = C_3C_2^* + C_3^*C_2 - ik_0(C_1C_2^* - C_1^*C_2)(\zeta')^2 - ik_0\rho^{-1}(C_3C_1^* - C_3^*C_1)\zeta' - (\omega^2/\rho c^2)|C_1|^2\zeta' - \rho|C_2|^2\zeta' \quad (35)$$

Here C_1, C_2, C_3 denote the three quantities F, $(F_z - ik_0F\zeta') + (k_0^2/\rho)|C_1|^2\,[\zeta' - (\zeta')^3]$, and $F_r + \zeta' F_z$ that are continuous at the interface. [See Eq (13).] Those terms in Eq. (35) that have no discontinuous factors make no contribution to the difference across the interface, so some additional simplification results, with Eq. (34) now taking the form

$$\frac{dk_0^2}{dr}\int |F|^2\rho^{-1}\,dz = P\int\left[\left(\frac{\omega^2}{c^2}\right)_r |F|^2\rho^{-1} - \frac{\rho}{2}\left(\frac{1}{\rho}\right)_r \frac{\partial K}{\partial z}\right]dz + k_0^2|C_1|^2\,[\zeta' - (\zeta')^3]\Delta(1/\rho) \times \Delta(1/\rho) - |C_1|^2\zeta'\,\Delta\left(\frac{\omega^2}{\rho c^2}\right) - |C_2|^2\zeta'\,\Delta\rho - 2k_0 Im(C_1C_3^*)\zeta'\,\Delta\left(\frac{1}{\rho}\right), \quad (36)$$

where the symbol Δ denotes the difference of the value on larger-z side minus the value on smaller-z side.

Equation (36) confirms the assertion that k_0 should be constant if the environment is range independent, since all the terms vanish identically if c and ρ are independent of r and if there is no sloping interface at which either are discontinuous. The equation should also give some insight into whether k_0 should increase or decrease when the environment varies with range. For example, it implies, when the overall sound speed tends to decrease with increasing range, that this trend tends to cause k_0 to increase. If a shallow-water layer of gradually decreasing depth overlies a fluid bottom with greater sound speed such that ζ' is negative and $\Delta(1/c^2)$ is negative, then there is a tendency for k_0 to decrease with increasing range. In general, however, because some of the terms have competing signs, one needs some rudimentary knowledge of the depth structure of the wavefield to make definitive assertions in this regard. In any event the result should be helpful in checking the validity of numerical computations.

In routine numerical computations for propagation in range-dependent media, given that the initial value of k_0 has been set with Eq. (24), the simplest procedure for determining k_0 at each successive integration step would appear to be that of using the conservation of energy relation (16).

7. CONNECTION WITH MODE THEORY

Because of Fitzgerald's[15] and McDaniel's[16] observation, mentioned in the introduction of this paper, that a parabolic approximation should be exact for single-guided-mode propagation in a range-independent environment if k_0 is taken to be the modal wavenumber, it is of interest to learn what the k_0 derived in the present paper would be for such circumstances. Since the derivation of the preceding section shows that k_0 must be constant when the environment is range independent, one can solve the parabolic equation (9) by the same techniques (separation of variables and superposition) that are used in deriving the modal solution of Eq. (3). In such a manner one finds

$$F = \sum_n A_n e^{i\gamma_n r} Z_n(z), \tag{37}$$

$$\gamma_n = (k_n^2 - k_0^2)/2k_0, \tag{38}$$

where the A_n are the same modal coefficients as appear in Eqs. (19); the $Z_n(z)$ and k_n^2 are the eigenfunctions and eigenvalues of the differential equation (20). Because this differential equation is of Sturm–Liouville type, one can assume with no loss of generality that all of the $Z_n(z)$ are real.

That $k_0 = k_n$ for the single-mode case follows directly from the eigenvalue problem that corresponds to the differential equation (20). If one multiplies both sides of that equation by $\rho^{-1}Z_n(z)$, then integrates over depth, transforming the term involving the second derivative with respect to z with an integration by parts, then solves for k_n^2, the result is

$$k_n^2 = \frac{\int (\omega/c)^2 Z_n^2 \rho^{-1}\,dz - \int (Z_n')^2 \rho^{-1}\,dz}{\int Z_n^2 \rho^{-1}\,dz}. \tag{39}$$

[The derivation assumes that the boundary conditions on the ordinary differential equation are such that $Z_n Z_n'/\rho$ vanishes at the endpoints. Any physically acceptable fluid-dynamic model of guided underwater sound propagation will lead to such boundary conditions if guided waves are to exist.]

To show that $k_0 = k_n$ for single-mode propagation, one need only substitute

$$F = A_n Z_n(z) e^{i\gamma_n r} \tag{40}$$

into Eq. (24). The resulting right side is the same as the right side of Eq. (39), so the identification follows.

To check whether the natural k_0 adheres to Fitzgerald's suggestion that it should be an average (with some weighting) of the k_n's for the propagating modes, one inserts for F, into the right side of Eq. (24), the expression (37). A typical integrand, because it involves a square of a magnitude of a sum, becomes a double sum over, say, n and m. The order—summation then integration—can be interchanged, and one is confronted with integrals of the generic types

$$\int Z_n Z_m \rho^{-1}\,dz, \quad \int \left(\frac{\omega}{c}\right)^2 Z_n Z_m \rho^{-1}\,dz, \quad \int Z_n' Z_m' \rho^{-1}\,dz.$$

The third such integral can be transformed by an integration by parts, and then one can substitute for the second derivative factor using the original differential equation (20). Doing this allows one to discover that all the modal cross terms either mutually cancel or else vanish because of the orthogonality relation (22). The result is then simply that

$$k_0^2 = \sum k_n^2 |A_n|^2 \int Z_n^2 \rho^{-1}\,dz \Big/ \sum |A_n|^2 \int Z_n^2 \rho^{-1}\,dz, \tag{41}$$

so that k_0 is an rms-weighted average of the modal wave numbers, the weighting coefficients being the squared magnitudes of the modal amplitudes if all the eigenfunctions are normalized such that the integral over depth of $\rho^{-1}Z_n^2(z)$ is unity. One may note that the result (41) is wholly in accord with Fitzgerald's suggestion.

8. CONCLUDING REMARKS

The incorporation of the definition (24) for k_0^2 into existing PE computational algorithms is trivial if the environment is range-independent, since k_0 need only be computed at the initial range; for range-dependent environments the use of a k_0 that varies with r and which must be computed at successive range increments could add additional complexities and require more computation time. Runs on a number of test cases[34] at the Naval Underwater Systems Center indicate the use of Eq. (24) consistently yields more accurate results than when other schemes are used to select k_0; yet to be explored are cases where such other schemes would give substantially erroneous results. A possible example is when three modes are simultaneously propagating upslope in shallow water of gradually decreasing depth, the highest-order mode initially much more strongly excited but disappearing first at its cutoff depth.

Having a formulation in which there is a natural definition of a horizontal wave number at every range is strongly analogous to the adiabatic mode theory[35], and the formulation here does reduce to that theory if there is only one propagating mode and if the environment is slowly varying with range. This analogy suggests a simplified method for extending the parabolic approximation to three dimensions, where the environment depends on azimuth as well as on range: The range coordinate in the two-dimensional PE formulation is replaced by distance along a horizontal ray path; the horizontal gradients of $k_0(x, y)$ cause the horizontal ray path to refract; amplitudes grow or diminish, depending on whether adjacent paths are converging or spreading. This possibility has been briefly mentioned earlier[36], but remains to be developed and implemented.

Acknowledgements—The author thanks Ding Lee, Frederick Tappert and Robert Greene for their helpful comments.

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